INTERNATIONAL JOURNAL FOR NUMERICAL METHODS IN FLUIDS, VOL. 15, 959–965 (1992)

ESTIMATING RAINFALL–RUN-OFF MODEL PARAMETERS BY NON-LINEAR MINIMIZATION

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SUMMARY

Existing discrete, linear rainfall–run-off models generally require the effective rainfall of a given storm as the input for computing the run-off hydrograph. This paper proposes a rainfall–run-off model which uses the rainfall hyetograph as input and directly accounts for rainfall losses. The model combines an ARMA model and a modified Philip equation for rainfall losses due to infiltration. For a given watershed with measured rainfall hyetograph and the corresponding run-off hydrograph, optimal values of model parameters are estimated by using a non-linear iterative technique. Applications of the model to two different watersheds show that the computed run-off hydrographs agree well with the measurements. The proposed model is a viable alternative to the widely used unit-hydrograph method.

KEY WORDS Rainfall–run-off model Time series analysis Non-linear optimization

INTRODUCTION

Rainfall–run-off models generally are used to relate the effective rainfal of a storm and the resulting direct run-off for a given drainage area without directly involving the losses of rainfall in the model. However, in practical applications of such models the effective rainfall must be determined from the measured rainfall and estimated hydrologic abstractions such as infiltration, depression storage, interception, etc. The estimation of hydrologic abstractions contains a high degree of uncertainty which would directly affect the accuracy of the computed run-off hydrograph. The purpose of this paper is to extend the ARMA model used by Wang and Yu[1] to include explicitly the time-variant rainfall losses due to infiltration in the model. With the measured rainfall hyetograph and run-off hydrograph for a given storm, the optimal values of model parameters can be estimated by using a non-linear iterative technique. Conversely, with the estimated model parameters, the model can be used to calculate directly the run-off hydrograph for a given storm hyetograph. Applications of this model to two different watersheds show good agreement between the computed and measured run-off hydrographs.

0271–2091/92/210959–07$08.50

Received March 1992

RAINFALL–RUN-OFF MODEL WITH INFILTRATION LOSSES

General input–output model

For a continuous linear system the relationship between the time-dependent input $I(t)$ and output $Q(t)$ is described by the differential equation

$$\left(1+\sum_{i=1}^{p}\alpha_i D^i\right)Q(t)=\left(\sum_{j=1}^{q}\beta_j D^j\right)I(t), \tag{1}$$

where $I(t)$ is the input (rainfall excess), $Q(t)$ is the output (run-off), D^i is the ith-order differential operator and the α_i and β_j are parameters. In practice, discrete models are of general interest. The discrete form of (1) can be derived by way of its finite difference form as given by Box and Jenkins[2] in the form

$$(1-a_1B-a_2B^2-\ldots-a_pB^p)Q(t)=(b_0+b_1B+b_2B^2+\ldots+b_qB^q)I(t), \tag{2}$$

where B is the backward shift operator, $B[Q(t)]=Q(t-1)$.

Rainfall–run-off model with infiltration losses

The discrete, linear rainfall–run-off model given by Wang and Yu[1] can be extended to include rainfall losses in the following form for the direct run-off $Q(t)$ at time t:

$$Q(t)=a_1Q(t-1)+a_2Q(t-2)+\ldots+a_pQ(t-p)+b_0I(t)+b_1I(t-1)+\ldots+b_qI(t-q), \tag{3}$$

where $I(t)=P(t)-L(t)$ for $t=1, 2, \ldots, t-q$, $P(t)$ is the rainfall at time t and $L(t)$ is the rainfall loss at time t. For $t=1, 2, \ldots, m$, equation (3) can be written as

$$\begin{aligned}
Q(1)&=b_0[P(1)-L(1)]\\
Q(2)&=a_1Q(1)+b_0[P(2)-L(2)]+b_1[P(1)-L(1)]\\
&\vdots\\
Q(s)&=a_1Q(s-1)+a_2Q(s-2)+\ldots+a_pQ(s-p)+b_0[P(s)-L(s)]+\ldots+b_q[P(s-q)-L(s-q)]\\
&\vdots\\
Q(m)&=a_1Q(m-1)+a_2Q(M-2)+\ldots+a_pQ(m-p),
\end{aligned} \tag{4}$$

where s is the number of periods of measured hyetograph with $m>s+q$ and the as and bs are parameters of the ARMA (p, q) model. Adding all equations in (4) and dividing the resulting equation by $\Sigma Q(t)$ gives

$$\begin{aligned}
1=&\,a_1\left(\sum_{t=1}^{m-1}Q(t)\Big/\sum_{t=1}^{m}Q(t)\right)+\ldots+a_p\left(\sum_{t=1}^{m-p}Q(t)\Big/\sum_{t=1}^{m}Q(t)\right)\\
&+b_0\left(\sum_{t=1}^{m}[P(t)-L(t)]\Big/\sum_{t=1}^{m}Q(t)\right)+\ldots+b_q\left(\sum_{t=1}^{m-q}[P(t)-L(t)]\Big/\sum_{t=1}^{m}Q(t)\right).
\end{aligned} \tag{5}$$

In practical problems the number of time intervals p for the run-off hydrograph is generally much larger than q and the non-zero rainfalls occur only in the first few time intervals for a given storm. When m is taken large enough to encompass the entire run-off hydrograph, then from mass conservation, the coefficients of bs, the terms in large parentheses in equation (5) are all close to unity. Thus the following approximate relation between the as and bs holds:

$$a_1+a_2+\ldots+a_p+b_0+b_1+\ldots+b_q=1. \tag{6}$$

Equation (6) is a constraint on the parameters of the ARMA (p, q) model.

Modified Philip equation

The empirical infiltration equation given by Philip[3] is

$$f(t)=At^{-1/2}+B, \tag{7}$$

where $f(t)$ is the infiltration capacity at time t and A and B are empirical constants. Based on extensive field experiments conducted by the Institute of Geography, Academia Sinica, equation (7) has been modified to the following form to include the effect of rainfall rate on the field infiltration rate:

$$f(t)=(At^{-1/2}+B)P(t)^{0.5} \tag{8}$$

where $P(t)$ is the rainfall intensity during period t. Integration of (8) from t to $t+1$ yields the loss due to infiltration,

$$L(t)=\int_t^{t+1} f(t)\mathrm{d}t=\{A[(t+1)^{1/2}-t^{1/2}]/2+B\}P(t)^{1/2}. \tag{9}$$

Substituting equation (9) into equation (4) gives a set of equations involving two more parameters A and B in addition to the as and bs of the ARMA (p, q) model. The estimation of these parameters is described in the following section.

Table I. Measured and computed run-off hydrographs for Shoal Creek, Texas

Estimated model parameters

$a_1=0{\cdot}49$, $a_2=0{\cdot}87$, $a_3=-0{\cdot}37$, $b_0=-0{\cdot}01$, $b_1=0{\cdot}60$, $b_2=-0{\cdot}58$, $A=45{\cdot}5$, $B=0{\cdot}017$

Run-off hydrograph

Time (h:min)	Rainfall (mm)	Run-off (cm) Measured	Run-off (cm) Computed
19:20	300·6	0·3	0·0
19:50	82·2	10·5	7·2
20:20	78·4	12·5	7·6
20:50	172·1	14·3	16·3
21:20	140·0	59·8	60·6
22:20	25·7	30·5	31·1
22:50	0·0	12·2	12·4
23:20	0·0	6·4	6·3
23:50	0·0	1·8	2·1
0:20	0·0	1·1	0·8
0:50	0·0	0·8	1·4
1:20	0·0	0·6	0·6
1:50	0·0	0·5	1·2

ESTIMATION OF MODEL PARAMETERS

Let the unknown parameters be represented by a column vector

$$\mathbf{X}=[a_1,\ldots,a_p;b^0,b_1,\ldots,b_q;A,B]^{\mathrm{T}}, \tag{10}$$

where T designates the transpose of the row vector. The optimal estimate of **X** can be obtained by minimizing the sum of squares of differences between the measured and computed outputs. Let $u(\mathbf{X})$ be the difference between the measured run-off $Q(t)$ and the computed run-off $\hat{Q}(t)$ at time t; then

$$\phi_t(\mathbf{X})=\hat{Q}(t)-Q(t) \quad \text{for } t=1,2,\ldots,m. \tag{11}$$

The optimal estimate of **X** is the one that minimizes the function

$$f(\mathbf{X})=\Sigma\phi_t^2(\mathbf{X})=\mathbf{\Phi}^{\mathrm{T}}(\mathbf{X})\mathbf{\Phi}(\mathbf{X}), \tag{12}$$

where $\mathbf{\Phi}(\mathbf{X})=[\phi_1(\mathbf{X}),\phi_2(\mathbf{X}),\ldots,\phi_m(\mathbf{X})]^{\mathrm{T}}$. The modified Gauss–Newton method also known as the Marquardt method[4] is used to find iteratively the optimal value of **X** as follows.

The gradient of the objective function, equation (12), may be expressed as

$$\nabla f(\mathbf{X})=2\mathbf{J}^{\mathrm{T}}(\mathbf{X})\mathbf{\Phi}(\mathbf{X}),$$

Table II. Measured and computed run-off hydrographs for Sanchuankon River, China

Estimated parameters

$a=1{\cdot}01$, $a=-0{\cdot}35$, $a=0{\cdot}114$, $b=-0{\cdot}003$, $b=0{\cdot}019$, $b=0{\cdot}213$, $A=18{\cdot}5$, $B=0{\cdot}483$

Run-off hydrograph

Time (h:min)	Rainfall (mm)	Run-off (cm)	
		Measured	Computed
19:00	308·0	0·0	0·0
19:05	442·6	0·0	0·0
19:10	299·4	4·9	6·1
19:15	42·5	73·6	71·7
19:20	0·0	109·7	112·8
19:25	0·0	94·2	91·3
19:30	0·0	61·8	60·6
19:35	0·0	39·0	41·9
19:40	0·0	30·7	31·3
19:45	0·0	22·0	23·8
19:50	0·0	18·1	17·7
19:55	0·0	14·2	13·0
20:00	0·0	10·3	9·7
20:05	0·0	7·7	7·2
20:10	0·0	5·9	5·3
20:15	0·0	4·8	3·9
20:20	0·0	4·0	2·9
20:25	0·0	3·2	2·2

where $\mathbf{J}(\mathbf{X})$ is the $m \times n$ Jacobian matrix and $n = p + q + 1$:

$$\mathbf{J}(\mathbf{X}) = \begin{bmatrix} \partial\phi_1/\partial x_1 & \partial\phi_1/\partial x_2 & \cdots & \partial\phi_1/\partial x_n \\ . & . & \cdots & . \\ \partial\phi_m/\partial x_1 & \partial\phi_m/\partial x_2 & \cdots & \partial\phi_m/\partial x_n \end{bmatrix}. \tag{13}$$

Let $\mathbf{X}^{(k+1)}$ be the $(k+1)$th iteration value of $\mathbf{X}$. From equation (12) the gradient at $\mathbf{X}^{(k+1)}$ is approximated by

$$\nabla f(\mathbf{X}^{(k+1)}) = 2\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k+1)})\mathbf{\Phi}(\mathbf{X}^{(k+1)}) \tag{14}$$

and $\mathbf{\Phi}(\mathbf{X}^{(k+1)})$ is approximated by a first-order Taylor series

$$\mathbf{\Phi}(\mathbf{X}^{(k+1)}) = \mathbf{\Phi}(\mathbf{X}^{(k+1)}) + \mathbf{J}(\mathbf{X}^{(k)})\,(\mathbf{X}^{(k+1)} - \mathbf{X}^{(k)}). \tag{15}$$

Then using (15) in (13) gives an approximation for the gradient of $f(\mathbf{X})$ at the new point $\mathbf{X}^{(k+1)}$.

The necessary condition for a minimum to exist at $\mathbf{X}^{(k+1)}$ is

$$\nabla f(\mathbf{X}^{(k+1)}) = 0. \tag{16}$$

Equations (14)–(16) lead to the Gaussian least-squares method for the approximation of $\mathbf{X}^{(k+1)}$:

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - [\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k)})\,\mathbf{J}(\mathbf{X}^{(k)})]^{-1}\,\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k)})\,\mathbf{\Phi}(\mathbf{X}^{(k)}). \tag{17}$$

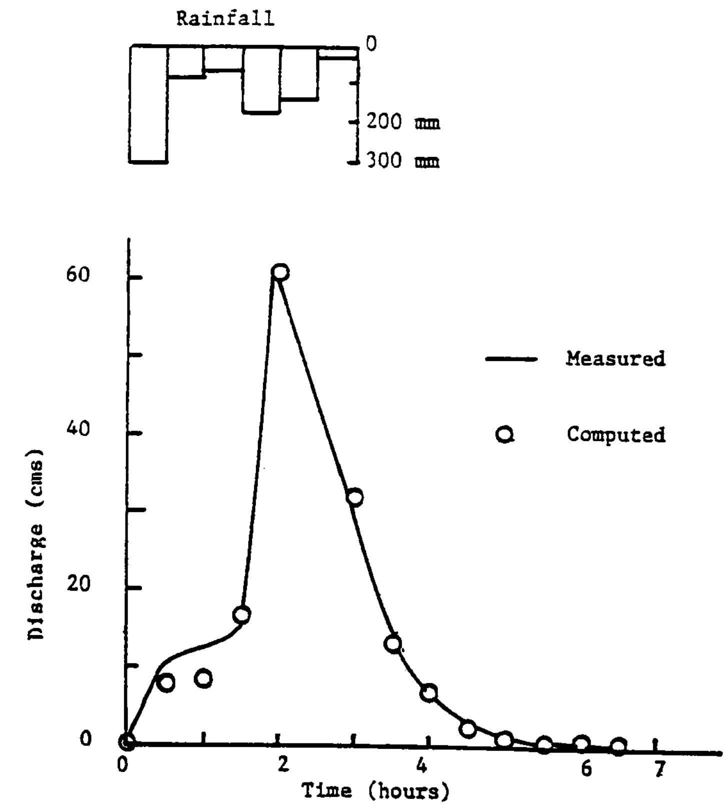

Figure 1. Measured and computed run-off hydrographs for Shoal Creek, Texas (19–20 July 1979, drainage area 18·2 km^2)

The Marquardt correction of equation (17) takes the form

$$\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} - [\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k)})\,\mathbf{J}(\mathbf{X}^{(k)}) + \beta\mathbf{I}]^{-1}\,\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k)})\,\mathbf{\Phi}(\mathbf{X}^{(k)}), \tag{18}$$

where β is a positive constant such that β is larger than the absolute value of the minimum eigenvalue of $\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k)})\ \mathbf{J}(\mathbf{X}^{(k)})$ and $\mathbf{I}$ is an identity matrix. The Marquardt method forces the Hessian matrix to be positive definite at each stage of the minimization and ensures that the

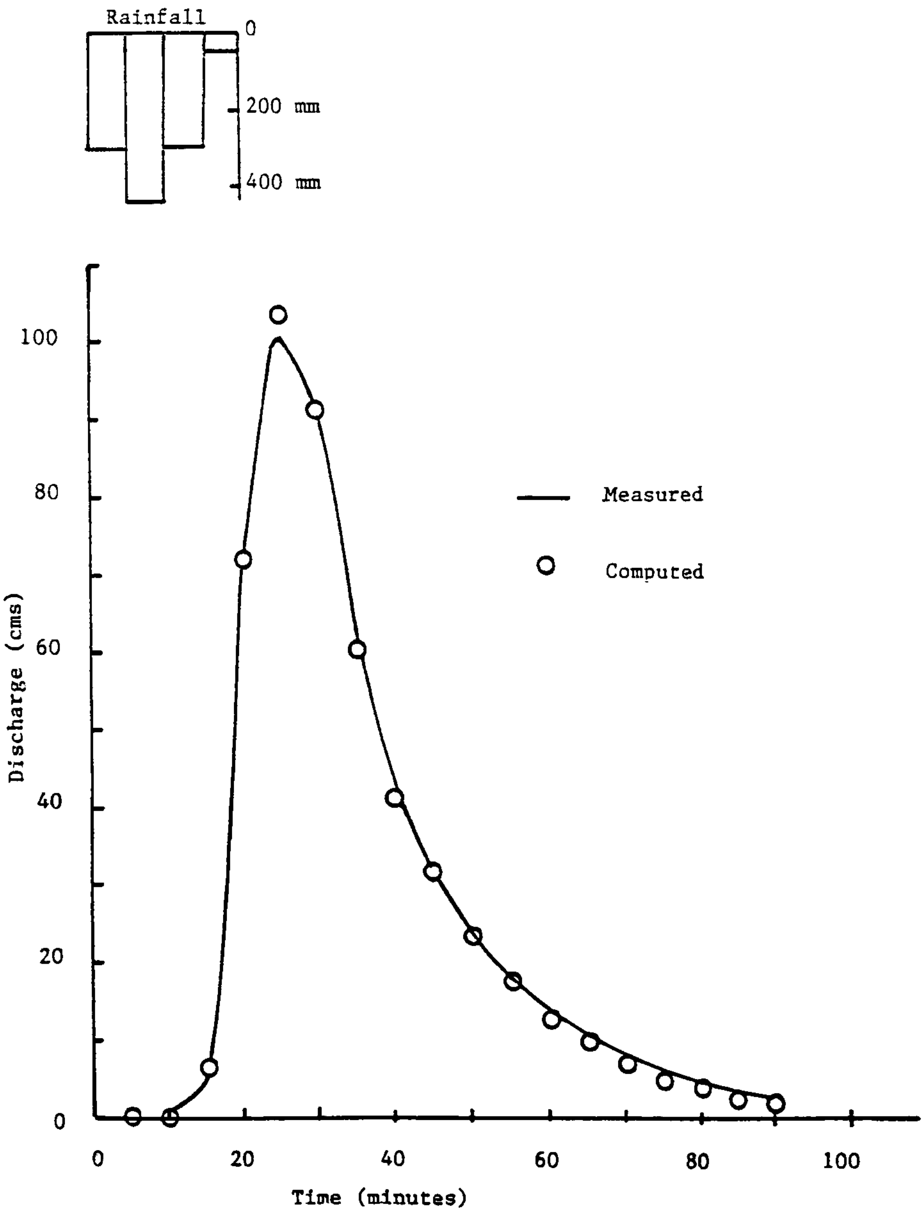

Figure 2. Measured and computed run-off hydrographs for Sanchuankon River, China (28 August 1966, drainage area 21·0 km^2)

estimate of the inverse of the Hessian matrix is positive definite. This method is found to be robust by Bard (1970) in the estimation of a relatively large number of parameters.

The iterative procedure is as follows.

1. At the kth iteration with $\mathbf{X}^{(k)}$ known from the preceding step, evaluate $\mathbf{J}(\mathbf{X}^{(k)})$, equation (13), and compute $\mathbf{J}^{\mathrm{T}}(\mathbf{X}^{(k)})\mathbf{J}(\mathbf{X})^{(k)}) + \beta\mathbf{I}$ and $f(\mathbf{X}^{(k)})$.
2. Compute $\mathbf{X}^{(k+1)}$ from equation (17).
3. Return to steps 1 and 2 until the specified termination criterion is satisfied.

APPLICATIONS

The method was applied to two relatively small watersheds: one is the Shoal Creek (18·2 km^2) in Texas and the other the Sanchuankon River (21·0 km^2) in China. The recorded rainfall and run-off data for the storm of 19–20 July 1979 for Shoal Creek were taken from Reference 6. The storm and run-off records for the 28 August 1966 storm were used for Sanchuankou River. For both cases an ARMA (3, 2) model was used with eight model parameters to be estimated for each watershed. The estimated values of model parameters and the computed and measured run-off hydrographs are tabulated in Tables I and II and plotted in Figures 1 and 2 respectively for Shoal Creek and Sanchuankou River. The computed and measured run-off hydrographs are generally in good agreement.

CONCLUSIONS

A discrete, linear rainfall–run-off model including infiltration losses is proposed. The model combines an ARMA (3, 2) model with a modified Philip equation for infiltration. With the measured rainfall hyetograph and run-off hydrograph, the model parameters can be estimated by using the Marquardt iterative optimization technique. Applications of the model to two different, relatively small watersheds demonstrate that the results are sufficiently accurate. The model has an apparent advantage over the unit-hydrograph technique because the measured hyetograph is used directly in the model and thus the uncertainty in estimating the effective rainfall is avoided. With model parameters determined, the model can be used to compute the run-off hydrograph directly from the measured rainfall hyetograph of a given storm. However, further research is needed to incorporate the initial soil moisture content in the model.

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